PARTIALLY ISOMETRIC TOEPLITZ OPERATORS ON THE POLYDISC

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Abstract. A Toeplitz operator T_φ, $\varphi \in L^\infty(\mathbb{T}^n)$, is a partial isometry if and only if there exist inner functions $\varphi_1, \varphi_2 \in H^\infty(\mathbb{D}^n)$ such that φ_1 and φ_2 depends on different variables and $\varphi = \bar{\varphi}_1\varphi_2$. In particular, for $n = 1$, along with new proof, this recovers a classical theorem of Brown and Douglas.

We also prove that a partially isometric Toeplitz operator is hyponormal if and only if the corresponding symbol is an inner function in $H^\infty(\mathbb{D}^n)$. Moreover, partially isometric Toeplitz operators are always power partial isometry (following Halmos and Wallen), and hence, up to unitary equivalence, a partially isometric Toeplitz operator with symbol in $L^\infty(\mathbb{T}^n)$, $n > 1$, is either a shift, or a co-shift, or a direct sum of truncated shifts. Along the way, we prove that T_φ is a shift whenever φ is inner in $H^\infty(\mathbb{D}^n)$.

1. Introduction

Toeplitz operators are one of the most useful and prevalent objects in matrix theory, operator theory, operator algebras, and its related fields. For instance, Toeplitz operators provide some of the most important links between index theory, C^*-algebras, function theory, and non-commutative geometry. See the monograph by Higson and Roe [14] for a thorough presentation of these connections, and consult the paper by Axler [2] for a rapid introduction to Toeplitz operators.

Evidently, a lot of work has been done in the development of one variable Toeplitz operators, and it is still a subject of very active research, with an ever-increasing list of connections and applications. But on the other hand, many questions remain to be settled in the several variables case, and more specifically in the open unit polydisc case (however, see [7, 8, 11, 17, 22]). The difficulty lies in the obvious fact that the standard (and classical) single variable tools are either unavailable or not well developed in the setting of polydisc. Evidently, advances in Toeplitz operators on the polydisc have frequently resulted in a number of new tools and techniques in operator theory, operator algebras, and related fields.

Our objective of this paper is to address the following basic question: Characterize partially isometric Toeplitz operators on $H^2(\mathbb{D}^n)$, where $H^2(\mathbb{D}^n)$ denotes the Hardy space over the unit polydisc $\mathbb{D}^n$. Recall that a partial isometry [12] is a bounded linear operator whose restriction to the orthogonal complement of its null space is an isometry.

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Before we answer the above question, we first recall that $H^2(\mathbb{D}^n)$ is the Hilbert space of all analytic functions f on $\mathbb{D}^n$ such that

$$\|f\| := \Big(\sup_{0 \leq r < 1} \int_{\mathbb{T}^n} |f(rz_1, \ldots, rz_n)|^2 d\boldsymbol{m}(z) \Big)^{\frac{1}{2}} < \infty,$$

where $d\boldsymbol{m}(z)$ is the normalized Lebesgue measure on the n-torus $\mathbb{T}^n$, and $z = (z_1, \ldots, z_n)$. We denote by $L^2(\mathbb{T}^n)$ the Hilbert space $L^2(\mathbb{T}^n, d\boldsymbol{m}(z))$. From the radial limits of square summable analytic functions point of view [20], one can identify $H^2(\mathbb{D}^n)$ with a closed subspace $H^2(\mathbb{T}^n)$ of $L^2(\mathbb{T}^n)$. Let $L^\infty(\mathbb{T}^n)$ denote the standard C^*-algebra of $\mathbb{C}$-valued essentially bounded Lebesgue measurable functions on $\mathbb{T}^n$. The *Toeplitz operator* T_φ with symbol $\varphi \in L^\infty(\mathbb{T}^n)$ is defined by

$$T_\varphi f = P_{H^2(\mathbb{D}^n)}(\varphi f) \qquad (f \in H^2(\mathbb{D}^n)),$$

where $P_{H^2(\mathbb{D}^n)}$ denotes the orthogonal projection from $L^2(\mathbb{T}^n)$ onto $H^2(\mathbb{D}^n)$. Also recall that

$$H^\infty(\mathbb{D}^n) = L^\infty(\mathbb{T}^n) \cap H^2(\mathbb{D}^n),$$

where $H^\infty(\mathbb{D}^n)$ denotes the Banach algebra of all bounded analytic functions on $\mathbb{D}^n$. A function $\varphi \in H^\infty(\mathbb{D}^n)$ is called *inner* if φ is unimodular on $\mathbb{T}^n$.

The answer to the above question is contained in the following theorem:

Theorem 1.1. *Let φ be a nonzero function in $L^\infty(\mathbb{T}^n)$. Then T_φ is a partial isometry if and only if there exist inner functions $\varphi_1, \varphi_2 \in H^\infty(\mathbb{D}^n)$ such that φ_1 and φ_2 depends on different variables and*

$$T_\varphi = T_{\varphi_1}^* T_{\varphi_2}.$$

In particular, if $n = 1$, then the only nonzero Toeplitz operators that are partial isometries are those of the form T_φ and T_φ^*, where $\varphi \in H^\infty(\mathbb{D})$ is an inner function. This was proved by Brown and Douglas in [5]. Actually, as we will see soon in this case that T_φ is not only an isometry but a shift.

A key ingredient in the proof of the Brown and Douglas theorem is the classical Beurling theorem [3]. Recall that the Beurling theorem connects inner functions in $H^\infty(\mathbb{D})$ with shift invariant subspaces of $H^2(\mathbb{D})$. However, in the present case of higher dimensions, this approach does not work, as is well known, Beurling type classification does not hold for shift invariant subspaces of $H^2(\mathbb{D}^n)$, $n > 1$ (however, see the proof of Theorem 4.1). Here, we exploit more analytic and geometric structure of $H^2(\mathbb{D}^n)$ and $L^2(\mathbb{T}^n)$ to achieve the main goal. Section 3 contains the proof of the above theorem.

Along the way to the proof of Theorem 1.1, in Section 2 we prove some basic properties of Toeplitz operators on the polydisc. Some of these observations are perhaps known (if not readily available in the literature) to experts, but they are necessary for our purposes here. We also remark that the proof of $\|T_\varphi\| = \|\varphi\|_\infty$, $\varphi \in L^\infty(\mathbb{T}^n)$, in Proposition 2.2 seems to be different even in the case of $n = 1$, as it avoids the standard techniques of the spectral radius formula (see Brown and Halmos [6, page 99] and the monographs [9, 18, 19]).

Moreover, in Section 4, we prove the following result, which connects inner functions with shift operators, and is also of independent interest: *If $\varphi \in H^\infty(\mathbb{D}^n)$ is a nonconstant inner function, then M_φ is a shift.*

Here, and in what follows, M_φ denotes the analytic Toeplitz operator T_φ whenever $\varphi \in H^\infty(\mathbb{D}^n)$. In this case, M_φ is simply the standard multiplication operator on $H^2(\mathbb{D}^n)$, that is, $M_\varphi f = \varphi f$ for all $f \in H^2(\mathbb{D}^n)$.

In Section 5, as a first application to Theorem 1.1, we classify partially isometric hyponormal Toeplitz operators. Recall that a bounded linear operator T on some Hilbert space is called hyponormal if $T^*T - TT^* \geq 0$. In Corollary 5.1, we prove the following: If T_φ, $\varphi \in L^\infty(\mathbb{T}^n)$, is a partial isometry, then T_φ is hyponormal if and only if φ is an inner function in $H^\infty(\mathbb{D}^n)$.

Secondly, following the Halmos and Wallen [13] notion of power partial isometries (also see an Huef, Raeburn and Tolich [1]), in Corollary 5.2 we prove that partially isometric Toeplitz operators are always power partial isometry. In Theorem 5.3, we further exploit the Halmos and Wallen models of power partial isometries, and obtain a connection between partially isometric Toeplitz operators, shifts, co-shifts, and direct sums of truncated shifts.

Finally, collecting all these results together, from an operator theoretic point of view, we obtain the following refinement of Theorem 1.1:

Suppose T_φ, $\varphi \in L^\infty(\mathbb{T}^n)$, is partially isometric. Then, up to unitary equivalence, T_φ is either a shift, or a co-shift, or a direct sum of truncated shifts.

We stress that the latter possibility is only restricted to the $n > 1$ case.

2. Preparatory results

In this section, we develop the necessary tools leading to the proof of Theorem 1.1. In this respect, we again remark that in what follows, we will often identify (via radial limits) $H^2(\mathbb{D}^n)$ with $H^2(\mathbb{T}^n)$ without further explanation. Given $\varphi \in L^\infty(\mathbb{T}^n)$, we denote by L_φ the *Laurent operator* on $L^2(\mathbb{T}^n)$, that is, $L_\varphi f = \varphi f$ for all $f \in L^2(\mathbb{T}^n)$. Note that

$$\|L_\varphi\|_{\mathcal{B}(L^2(\mathbb{T}^n))} = \|\varphi\|_\infty,$$

where $\|\varphi\|_\infty$ denotes the essential supremum norm of φ. The Toeplitz operator T_φ with symbol $\varphi \in L^\infty(\mathbb{T}^n)$ is given by

$$T_\varphi = P_{H^2(\mathbb{D}^n)} L_\varphi|_{H^2(\mathbb{D}^n)}.$$

Clearly, $T_\varphi \in \mathcal{B}(H^2(\mathbb{D}^n))$. Also note that a function $f = \sum\limits_{k \in \mathbb{Z}^n} a_k z^k \in L^2(\mathbb{T}^n)$ is in $H^2(\mathbb{D}^n)$ if and only if $a_k = 0$ whenever at least one of the k_j, $j = 1, \ldots, n$, in $k = (k_1, \ldots, k_n)$ is negative.

We start with a several variables analogue of brother Riesz theorem. We denote the set of zeros of a scalar-valued function f by $\mathcal{Z}(f)$.

Lemma 2.1. *If $f \in H^2(\mathbb{D}^n)$ is nonzero, then $\boldsymbol{m}(\mathcal{Z}(f)) = 0$.*

Proof. Let m denote the normalized Lebesgue measure on $\mathbb{T}$. Suppose f is a nonzero function in $H^2(\mathbb{D}^2)$. For w_1 and w_2 in $\mathbb{T}$ a.e., we define the slice functions f_{w_1} and f_{w_2} by $f_{w_1}(z) = f(w_1, z)$ and $f_{w_2}(z) = (z, w_2)$ for all $z \in \mathbb{T}$. Set

$$\mathcal{Z} = \{w_2 \in \mathbb{T} : f_{w_2} \equiv 0\}.$$

Note that $\mathcal{Z} \subseteq \mathcal{Z}(f_{w_1})$ for all $w_1 \in \mathbb{T}$. If $m(\mathcal{Z}) > 0$, then the classical brother Riesz theorem implies that f is identically zero. Therefore, $m(\mathcal{Z}) = 0$. Evidently

$$m(\mathcal{Z}(f_{w_2})) = \begin{cases} 1 & \text{if } w_2 \in \mathcal{Z} \\ 0 & \text{if } w_2 \in \mathcal{Z}^c, \end{cases}$$

and hence $w_2 \mapsto m(\mathcal{Z}(f_{w_2}))$ is a measurable function. By the Tonelli and Fubini theorem, we see that

$$\begin{aligned} (m \times m)(\mathcal{Z}(f)) &= \int_{\mathbb{T}} m(\mathcal{Z}(f_{z_2}))\, dm(z_2) \\ &= \int_{\mathcal{Z}} m(\mathcal{Z}(f_{z_2}))\, dm(z_2) + \int_{\mathcal{Z}^c} m(\mathcal{Z}(f_{z_2}))\, dm(z_2) \\ &= 0. \end{aligned}$$

The rest of the proof now follows easily by the induction on n. □

We refer to Rudin [20, Theorem 3.3.5] for a different proof of the above lemma (even in the context of functions in the Nevanlinna class). Also, see [23] for the same for functions in $H^\infty(\mathbb{D}^n)$. However, the present proof is direct and avoids the use of heavy machinery from function theory.

We now prove that $\|T_\varphi\|_{\mathcal{B}(H^2(\mathbb{D}^n))} = \|\varphi\|_\infty$. As we have pointed out already in the introductory section above, this may be known to experts. However, even when $n = 1$, the present proof seems to be direct as it avoids the standard techniques of the spectral radius formula. For instance, see the classic monograph [9, Corollary 7.8] and the recent monograph [18, Corollary 3.3.2].

Proposition 2.2. $\|T_\varphi\| = \|\varphi\|_\infty$ *for all* $\varphi \in L^\infty(\mathbb{T}^n)$*.*

Proof. Let $\mathcal{L}$ denote the set of Laurent polynomials in n variables. We compute

$$\begin{aligned} \|T_\varphi\| &= \sup\{|\langle \varphi f, g\rangle| : f, g \in H^2(\mathbb{D}^n), \|f\|, \|g\| \leq 1\} \\ &= \sup\{|\langle \varphi f, g\rangle| : f, g \in \mathbb{C}[z_1, \ldots, z_n], \|f\|, \|g\| \leq 1\} \quad \text{(by density of polynomials)} \\ &= \sup\{|\langle \varphi f, g\rangle| : f, g \in \mathcal{L}, \|f\|, \|g\| \leq 1\} \\ &= \|L_\varphi\| \\ &= \|\varphi\|_\infty. \end{aligned}$$

Note the third equality follows because any Laurent polynomial can be multiplied by a monomial to put it into polynomials. This completes the proof of the proposition. □

The above elegant proof is due to Professor Greg Knese and replaces our original proof, which was longer and technical.

Before proceeding to the proof of the main theorem, we conclude this section with a result concerning unimodular functions in $L^\infty(\mathbb{T}^n)$.

Corollary 2.3. *Suppose* φ *is a nonzero function in* $L^\infty(\mathbb{T}^n)$*. If* $\|T_\varphi f\| = \|\varphi\|_\infty \|f\|$ *for some nonzero* $f \in H^2(\mathbb{D}^n)$*, then* $\frac{1}{\|\varphi\|_\infty}\varphi$ *is unimodular in* $L^\infty(\mathbb{T}^n)$*.*

Proof. In view of Proposition 2.2, without loss of generality we may assume that $\|T_\varphi\| = 1$. Then

$$\int_{\mathbb{T}^n} |\varphi(z)|^2 |f(z)|^2 d\boldsymbol{m}(z) = \int_{\mathbb{T}^n} |f(z)|^2 d\boldsymbol{m}(z).$$

By Lemma 2.1, $|\varphi(z)| = 1$ for all $z \in \mathbb{T}^n$ a.e. and the result follows. $\square$

In particular, if T_φ, $\varphi \in L^\infty(\mathbb{T}^n)$, is a partial isometry, then φ is unimodular.

3. Proof of Theorem 1.1

In this section, without explicitly mentioning it in each instance, we always assume that T_φ, $\varphi \in L^\infty(\mathbb{T}^n)$, is partially isometric. Also, we frequently make use of the identification $H^2(\mathbb{D}^n) \cong H^2(\mathbb{T}^n)$ without mentioning it (see Section 2).

For simplicity we denote by $\mathcal{R}(T)$ the range of a bounded linear operator T. Clearly, $\mathcal{R}(T_\varphi)$ is a closed subspace of $H^2(\mathbb{D}^n)$.

Lemma 3.1. *$\mathcal{R}(T_\varphi)$ is invariant under M_{z_i}, $i = 1, \ldots, n$.*

Proof. Note that, since $\|T_\varphi\| = 1$, we have $\|\varphi\|_\infty = 1$. Suppose $f \in \mathcal{R}(T_\varphi)$. By Corollary 2.3, it follows that φ is unimodular, and hence $\|L_{\bar\varphi} f\| = \|f\|$. Since T_φ^* is an isometry on $\mathcal{R}(T_\varphi)$, we have

$$\|f\| = \|T_\varphi^* f\| \le \|L_{\bar\varphi} f\| = \|\bar\varphi f\| = \|f\|.$$

Therefore, $\|P_{H^2(\mathbb{D}^n)}(\bar\varphi f)\| = \|\bar\varphi f\|$, that is, $P_{H^2(\mathbb{D}^n)}(\bar\varphi f) = \bar\varphi f$. This implies that

$$\bar\varphi f \in H^2(\mathbb{D}^n), \tag{3.1}$$

and hence $z_i \bar\varphi f \in H^2(\mathbb{D}^n)$ for all $i = 1, \ldots, n$. Then

$$T_\varphi T_\varphi^*(z_i f) = T_\varphi(\bar\varphi z_i f) = P_{H^2(\mathbb{D}^n)}(|\varphi|^2 z_i f) = P_{H^2(\mathbb{D}^n)}(z_i f) = z_i f,$$

implies that $z_i f \in \mathcal{R}(T_\varphi)$ for all $i = 1, \ldots, n$. This completes the proof. $\square$

In what follows, if $i \in \{1, \ldots, n\}$ and k_i is a negative integer, then we write $z_i^{k_i} = \bar z_i^{-k_i}$.

Lemma 3.2. *For each $i = 1, \ldots, n$, the function φ cannot depend on both z_i and $\bar z_i$ variables at a time.*

Proof. We shall prove this by contradiction. Assume without loss of generality that φ depends on both z_1 and $\bar z_1$. Then

$$\varphi = \sum_{k=1}^{\infty} \bar z_1^k \varphi_{-k} \oplus \sum_{k=0}^{\infty} z_1^k \varphi_k,$$

and $\varphi_{-k_0} \neq 0$ for some $k_0 \neq 0$. Here $\varphi_k \in L^2(\mathbb{T}^{n-1})$, $k \in \mathbb{Z}$, is a function of $\{z_i, \bar z_j : i, j = 2, \ldots, n\}$. There exist non-negative integers $k_2, \ldots, k_n$, and $l_2, \ldots, l_n$ such that the coefficient of $\bar z_2^{k_2} \cdots \bar z_n^{k_n} z_2^{l_2} \cdots z_n^{l_n}$ in the expansion of the Fourier series of φ_{-k_0} is nonzero. Set

$$Z_{kl} := z_2^{k_2} \cdots z_n^{k_n} z_2^{l_2} \cdots z_n^{l_n},$$

and

$$f := T_\varphi(z_1^{k_0} Z_{kl}) - z_1 T_\varphi(z_1^{k_0 - 1} Z_{kl}).$$

Note that f is a nonzero function in $H^2(\mathbb{D}^n)$, and f does not depend on z_1. Since $T_\varphi(z_1^{k_0-1} Z_{kl}) \in \mathcal{R}(T_\varphi)$, Lemma 3.1 implies that $f \in \mathcal{R}(T_\varphi)$. In particular, by (3.1), $\bar\varphi f \in H^2(\mathbb{D}^n)$. On the other hand, since

$$\bar\varphi f = \sum_{k=1}^{\infty} z_1^k (f \bar\varphi_{-k}) \oplus \sum_{k=0}^{\infty} \bar z_1^k (f \bar\varphi_k),$$

it follows that $f\bar{\varphi}_k = 0$ for all $k > 0$. Since $\boldsymbol{m}(\{z \in \mathbb{T}^n : f(z) = 0\}) = 0$, we have $\bar{\varphi}_k = 0$ for all $k > 0$. This yields

$$\varphi = \sum_{k=0}^{\infty} \bar{z}_1^k \varphi_{-k},$$

and hence φ depends on $\bar{z}_1$ and does not depend on z_1. This is a contradiction. $\square$

We are now ready for the proof of Theorem 1.1.

Proof of Theorem 1.1. Suppose T_φ is a partial isometry. In view of Lemma 3.2, there exists a (possibly empty) subset C of $\{z_1, \ldots, z_n\}$ such that φ is analytic in z_i for all $z_i \in A := C^c$, and co-analytic in z_j for all $z_j \in C$. Let $A = \{z_{i_1}, \ldots, z_{i_p}\}$ and $C = \{z_{j_1}, \ldots, z_{j_q}\}$. Then $p + q = n$, and

$$\varphi = \sum_{k \in \mathbb{Z}_+^q} \bar{z}_C^k \varphi_{A,k},$$

where $\varphi_{A,k} \in H^2(\mathbb{D}^p)$ is a function of $\{z_{i_1}, \ldots, z_{i_p}\}$, $\bar{z}_C^k = \bar{z}_{j_1}^{k_1} \cdots \bar{z}_{j_q}^{k_q}$, and $k = (k_1, \ldots, k_q) \in \mathbb{Z}_+^q$. Note that

$$\varphi_{A,l} \in \mathcal{R}(T_\varphi) \qquad (l \in \mathbb{Z}_+^q).$$

Indeed, $\varphi_{A,0} = T_\varphi 1 \in \mathcal{R}(T_\varphi)$. Moreover, for each $l \in \mathbb{Z}_+^q \setminus \{0\}$, we have

$$T_\varphi z^l = P_{H^2(\mathbb{D}^n)}\Big(\sum_{k \in \mathbb{Z}_+^q} z_C^{l-k} \varphi_{A,k}\Big),$$

that is

$$T_\varphi z^l = \sum_{l-k \geq 0} z_C^{l-k} \varphi_{A,k}.$$

Here $l - k \geq 0$ means that $l_i - k_i \geq 0$ for all $i = 1, \ldots, q$. Thus the claim follows by induction. By (3.1), we have $\bar{\varphi}\varphi_{A,l} \in H^2(\mathbb{D}^n)$, $l \in \mathbb{Z}_+^q$. Therefore

$$\bar{\varphi}\varphi_{A,l} = \sum_{k \in \mathbb{Z}_+^q} z_C^k \overline{\varphi_{A,k}} \varphi_{A,l} \in H^2(\mathbb{D}^n) \qquad (l \in \mathbb{Z}_+^q).$$

Consequently, $\overline{\varphi}_{A,k}\varphi_{A,l} \in H^2(\mathbb{D}^p)$ for all k and l, and hence, in particular, we have

$$\overline{\varphi}_{A,l}\varphi_{A,l} \in H^2(\mathbb{D}^p) \qquad (l \in \mathbb{Z}_+^q).$$

This immediately implies that $\overline{\varphi}_{A,l}\varphi_{A,l}$ is a constant function, and hence $\varphi_{A,l} = \alpha_l \psi_l$ for some inner function $\psi_l \in H^\infty(\mathbb{D}^p)$ and scalar α_l such that $|\alpha_l| \leq 1$, $l \in \mathbb{Z}_+^q$. Assume without loss of generality that $\varphi_{A,0} \neq 0$. Now by the fact that $\overline{\varphi}_{A,0}\varphi_{A,k}$ and $\overline{\varphi}_{A,k}\varphi_{A,0}$ are in $H^2(\mathbb{D}^p)$, we have $\varphi_{A,k} = \beta_k \psi_0$, $k \in \mathbb{Z}_+^q$. Therefore

$$\varphi = \Big(\sum_{k \in \mathbb{Z}_+^q} \beta_k \bar{z}_C^k\Big)\psi_0 = \bar{\varphi}_1 \varphi_2,$$

where $\varphi_1 = \sum_{k \in \mathbb{Z}_+^q} \bar{\beta}_k z_C^k$ and $\varphi_2 = \psi_0$.

We now turn to the converse part. First we have clearly

$$T_{\varphi_1} T_{\varphi_2} = T_{\varphi_2} T_{\varphi_1}. \tag{3.2}$$

We also claim that

$$T_{\varphi_1} T_{\varphi_2}^* = T_{\varphi_2}^* T_{\varphi_1}. \tag{3.3}$$

This holds trivially when one of the functions φ_1 or φ_2 is constant. We continue with the above notation, and assume that both A and C are nonempty subsets of $\{z_1, \ldots, z_n\}$. First we observe that φ_1 and φ_2 depends only on $\{z_{i_1}, \ldots, z_{i_p}\}$ and $\{z_{j_1}, \ldots, z_{j_q}\}$, respectively. Consider a monomial $z^k \in \mathbb{C}[z_1, \ldots, z_n]$. Suppose $k = (k_1, \ldots, k_n)$, and write

$$z^k = z_C^{k_c} z_A^{k_a},$$

where $k_c = (k_{j_1}, \ldots, k_{j_q}) \in \mathbb{Z}_+^q$, and $k_a \in \mathbb{Z}_+^p$ is the ordered p tuple made out of $\{k_i\}_{i=1}^n \setminus \{k_{j_t}\}_{t=1}^q$. Since the analytic function φ_2 depends only on $z_{j_s} \in C$, $s = 1, \ldots, p$, it is clear that

$$\bar{\varphi}_2 z_C^{k_c} = \varphi_a + \varphi_c,$$

where φ_a depends only on $\{z_{j_s}\}_{s=1}^p$ (and hence it is an analytic function) and $\varphi_c \in L^2(\mathbb{T}^q) \ominus H^2(\mathbb{D}^q)$ is a function of $\{z_{j_t}, \bar{z}_{j_t}\}_{t=1}^q$. Note that the latter property ensures that $\varphi_c(0) = 0$. Then, on one hand, we have

$$T_{\varphi_2}^* T_{\varphi_1} z^k = P_{H^2(\mathbb{D}^n)}(\bar{\varphi}_2 \varphi_1 z^k) = P_{H^2(\mathbb{D}^n)}\Big((\varphi_a + \varphi_c)\varphi_1 z_A^{k_a}\Big) = \varphi_a \varphi_1 z_A^{k_a},$$

and on the other hand that

$$T_{\varphi_1} T_{\varphi_2}^* z^k = \varphi_1 P_{H^2(\mathbb{D}^n)}(\bar{\varphi}_2 z^k) = \varphi_1 P_{H^2(\mathbb{D}^n)}\Big((\varphi_a + \varphi_c) z_A^{k_a}\Big) = \varphi_1 \varphi_a z_A^{k_a}.$$

Consequently, $T_{\varphi_2}^* T_{\varphi_1} z^k = T_{\varphi_1} T_{\varphi_2}^* z^k$ for all $k \in \mathbb{Z}_+^n$, which proves our claim. Now suppose that $T_\varphi = T_{\varphi_1}^* T_{\varphi_2}$, where φ_1 and φ_2 depends on different variables. Using (3.2) and (3.3), we obtain

$$T_\varphi T_\varphi^* = T_{\varphi_1}^* T_{\varphi_2} T_{\varphi_2}^* T_{\varphi_1} = (T_{\varphi_1}^* T_{\varphi_1})(T_{\varphi_2} T_{\varphi_2}^*) = P_{\mathcal{R}(T_{\varphi_2})}, \tag{3.4}$$

which implies that T_φ is a partial isometry. □

We remark that the commutativity and doubly commutativity of T_{φ_1} and T_{φ_2} in (3.2) and (3.3) will be useful in the particular applications to Theorem 1.1 in the final section.

4. Inner functions and shifts

In this short section, we pause to prove an auxiliary result that is both a necessary tool for our final refinement of partial isometric Toeplitz operators and a subject of independent interest with its own applications.

Let $\varphi \in H^\infty(\mathbb{D}^n)$, and suppose the multiplication operator M_φ is an isometry on $H^2(\mathbb{D}^n)$. Then

$$\|\varphi\|_\infty = \|M_\varphi\|_{\mathcal{B}(H^2(\mathbb{D}^n))} = 1,$$

and hence Corollary 2.3 implies that φ is a unimodular function in $H^\infty(\mathbb{D}^n)$, that is, φ is an inner function. Now we prove that a nonconstant inner function always defines a shift (and not only isometry). Recall that an operator $V \in \mathcal{B}(\mathcal{H})$ is said to be a *shift* if V is an isometry and $V^{*m} \to 0$ as $m \to \infty$ in the strong operator topology.

Recall that a closed subspace $\mathcal{S} \subseteq H^2(\mathbb{D}^n)$ is of *Beurling type* if there exists an inner function $\theta \in H^\infty(\mathbb{D}^n)$ such that $\mathcal{S} = \theta H^2(\mathbb{D}^n)$. It is also known that (cf. [16, Corollary 6.3] and [15]) a closed subspace $\mathcal{S} \subseteq H^2(\mathbb{D}^n)$, $n > 1$, is of Beurling type if and only if

$R_i^*R_j = R_jR_i^*$ for all $1 \leq i < j \leq n$, where $R_p = M_{z_p}|_{\mathcal{S}} \in \mathcal{B}(\mathcal{S})$ is the restriction operator and $p = 1, \ldots, n$. Note that

$$R_i^*R_j = P_{\mathcal{S}}M_{z_i}^*M_{z_j}|_{\mathcal{S}} \text{ and } R_jR_i^* = M_{z_j}P_{\mathcal{S}}M_{z_i}^*|_{\mathcal{S}}, \tag{4.1}$$

for all $i, j = 1, \ldots, n$.

Theorem 4.1. *If $\varphi \in H^\infty(\mathbb{D}^n)$ is a nonconstant inner function, then M_φ is a shift.*

Proof. It is well known (as well as easy to see) that M_φ is an isometry. Following the classical von Neumann and Wold decomposition for isometries, we only need to prove that

$$\mathcal{H}_u := \bigcap_{m=0}^{\infty} \varphi^m H^2(\mathbb{D}^n) = \{0\}.$$

Assuming the contrary, suppose that $\mathcal{H}_u \neq \{0\}$. We claim that $\mathcal{H}_u$ is of Beurling type. Since the $n = 1$ case is obvious, we assume that $n > 1$. As $\varphi^p H^2(\mathbb{D}^n) \subseteq \varphi^q H^2(\mathbb{D}^n)$ for all $p \geq q$, we have

$$P_{\mathcal{H}_u} = SOT - \lim_{m\to\infty} P_{\varphi^m H^2(\mathbb{D}^n)}.$$

Since $\varphi^m H^2(\mathbb{D}^n)$, $m \geq 1$, is a Beurling type invariant subspace, in view of (4.1), it follows that

$$P_{\mathcal{H}_u}M_{z_i}^*M_{z_j}h = M_{z_j}P_{\mathcal{H}_u}M_{z_i}^*h,$$

for all $h \in \mathcal{H}_u$. Then (4.1) again implies that $\mathcal{H}_u$ is of Beurling type. Therefore, there exists an inner function $\theta \in H^\infty(\mathbb{D}^n)$ such that $\mathcal{H}_u = \theta H^2(\mathbb{D}^n)$ (note that the $n = 1$ case directly follows from Beurling). Then, for each $m \geq 1$, there exists an inner function $\psi_m \in H^\infty(\mathbb{D}^n)$ such that $\theta = \varphi^m\psi_m$ (for instance, see (5.1)). Since φ is a nonconstant inner function, by the maximum modulus principle [21, §2, Theorem 6], we have $|\varphi(z)| < 1$ for all $z \in \mathbb{D}^n$. For each fixed $z_0 \in \mathbb{D}^n$, it follows that

$$|\theta(z_0)| = |\varphi(z_0)|^m|\psi_m(z_0)| \leq |\varphi(z_0)|^m \to 0 \text{ as } m \to \infty,$$

and hence $\theta \equiv 0$. This contradiction shows that $\mathcal{H}_u = \{0\}$. ☐

In fact, the above argument yields something more: Suppose $\{\mathcal{S}_m\}_{m\geq1}$ be a sequence of Beurling type invariant subspaces of $H^2(\mathbb{D}^n)$. Then $\bigcap_{m=1}^\infty \mathcal{S}_m$ is also a Beurling type invariant subspace. Indeed, we let $\mathcal{H}_m = \bigcap_{i=1}^m \mathcal{S}_m$. Then $\{\mathcal{H}_m\}_{m\geq1}$ forms a decreasing sequence of Beurling type invariant subspaces, and hence

$$P_{\bigcap_{m=1}^\infty \mathcal{S}_m} = P_{\bigcap_{m=1}^\infty \mathcal{H}_m} = SOT - \lim_{m\to\infty} P_{\mathcal{H}_m}.$$

The rest of the proof is then much as before.

We also wish to point out that Theorem 4.1 can be proved by using (analytic) reproducing kernel Hilbert space techniques. We believe that the algebraic tools described above might be useful in other settings.

5. Applications and further refinements

We begin with partially isometric Toeplitz operators that are hyponormal. A bounded linear operator T acting on a Hilbert space is called *hyponormal* if $[T^*, T] \geq 0$, where

$$[T^*, T] = T^*T - TT^*,$$

is the self commutator of T.

Now suppose T_φ, $\varphi \in L^\infty(\mathbb{T}^n)$, is a partial isometry. If $\varphi \in H^\infty(\mathbb{D}^n)$ is inner, then T_φ is an isometry and hence is hyponormal. For the converse direction, we note by Theorem 1.1 that $T_\varphi = T^*_{\varphi_1} T_{\varphi_2}$ for some inner functions φ_1 and φ_2 in $H^\infty(\mathbb{D}^n)$ which depends on different variables. If φ_1 is a constant function, then $T_\varphi = T_{\varphi_2} = M_{\varphi_2}$ is an isometry, and hence T_φ is hyponormal. If φ_2 is a constant function, then $T_\varphi = T^*_{\varphi_1} = M^*_{\varphi_1}$ is a co-isometry, and hence T_φ cannot be hyponormal. Suppose both φ_1 and φ_2 are nonconstant functions. Now (3.2) and (3.3) imply that

$$T^*_\varphi T_\varphi = T^*_{\varphi_2} T_{\varphi_1} T^*_{\varphi_1} T_{\varphi_2} = (T^*_{\varphi_2} T_{\varphi_2})(T_{\varphi_1} T^*_{\varphi_1}) = T_{\varphi_1} T^*_{\varphi_1}.$$

Then, by (3.4) we see that $[T^*_\varphi, T_\varphi] \geq 0$ implies $T_{\varphi_2} T^*_{\varphi_2} \leq T_{\varphi_1} T^*_{\varphi_1}$. By noting that φ_1 and φ_2 are analytic functions, we see

$$M_{\varphi_2} M^*_{\varphi_2} \leq M_{\varphi_1} M^*_{\varphi_1},$$

which, by the Douglas range inclusion theorem, is equivalent to $M_{\varphi_2} = M_{\varphi_1} X$ for some $X \in \mathcal{B}(H^2(\mathbb{D}^n))$. Observe that

$$M_{\varphi_1} M_{z_i} X = M_{z_i} M_{\varphi_1} X = M_{z_i} M_{\varphi_2} = M_{\varphi_2} M_{z_i} = M_{\varphi_1} X M_{z_i}, \tag{5.1}$$

implies that $M_{z_i} X = X M_{z_i}$ for all $i = 1, \ldots, n$, and hence $X = M_\psi$ for some $\psi \in H^\infty(\mathbb{D}^n)$. Hence, we conclude that $\varphi_2 = \varphi_1 \psi$. Since φ_1 and φ_2 are inner functions, $\psi \in H^\infty(\mathbb{D}^n)$ is inner. Moreover, since φ_1 and φ_2 depends on different variables, that $\varphi_2 = \varphi_1 \psi$ is possible if and only if ψ is a unimodular constant. Suppose $\varphi_2 = \alpha \varphi_1$, where $|\alpha| = 1$. Then

$$T_\varphi = T^*_{\varphi_1} T_{\varphi_2} = \alpha T^*_{\varphi_1} T_{\varphi_1} = \alpha T_{|\varphi_1|^2} = \alpha I_{H^2(\mathbb{D}^n)},$$

as $|\varphi_1|^2 = 1$ on $\mathbb{T}^n$, that is, T_φ is a unimodular constant times the identity operator. We have therefore shown the following result:

Corollary 5.1. *Let T_φ, $\varphi \in L^\infty(\mathbb{T}^n)$, be a partial isometry. Then T_φ is hyponormal if and only if φ is an inner function in $H^\infty(\mathbb{D}^n)$.*

Therefore, in view of Theorem 4.1, T_φ is hyponormal if and only if (up to unitary equivalence) T_φ is a shift.

We recall [13, Halmos and Wallen] that a bounded linear operator T acting on some Hilbert space is called a *power partial isometry* if T^m is partially isometric for all $m \geq 1$. Clearly, Theorem 1.1 and the equalities in (3.2) and (3.3) imply the following statement:

Corollary 5.2. *Partially isometric Toeplitz operators are power partial isometry.*

We also recall from Halmos and Wallen [13] (also see [1]) that every power partial isometry is a direct sum whose summands are unitary operators, shifts, co-shifts, and truncated shifts. Recall that a *truncated shift* S of *index* p, $p \in \mathbb{N}$, on some Hilbert space $\mathcal{H}$ is an operator of the form

$$S = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ I_{\mathcal{H}_0} & 0 & 0 & \cdots & 0 & 0 \\ 0 & I_{\mathcal{H}_0} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & I_{\mathcal{H}_0} & 0 \end{bmatrix}_{p \times p},$$

where $\mathcal{H}_0$ is a Hilbert space, and $\mathcal{H} = \underbrace{\mathcal{H}_0 \oplus \cdots \oplus \mathcal{H}_0}_{p}$.

We prove that, up to unitary equivalence, a partial isometric T_φ is simply direct sum of truncated shifts, or a shift, or a co-shift (that is, adjoint of a shift). The proof is essentially contained in Theorem 4.1 and the Halmos and Wallen models of power partial isometries.

Theorem 5.3. *Up to unitary equivalence, a partially isometric Toeplitz operator is either a shift, or a co-shift, or a direct sum of truncated shifts.*

Proof. Suppose T_φ, $\varphi \in L^\infty(\mathbb{T}^n)$, is a partial isometry. By Theorem 1.1, $T_\varphi = T^*_{\varphi_1} T_{\varphi_2}$, where φ_1 and φ_2 are inner functions in $H^\infty(\mathbb{D}^n)$ and depends on different variables. Moreover, by Corollary 5.2, T_φ is a power partial isometry. If φ_1 is a constant function, then T_φ is a shift, and if φ_2 is a constant function, then T_φ is a co-shift. Now let both φ_1 and φ_2 are nonconstant functions. Following the construction of Halmos and Wallen [13, page 660] (also see [1]), we set $E_m = T_\varphi^{*m} T_\varphi^m$ and $F_m = T_\varphi^m T_\varphi^{*m}$ for the initial and final projections of the partial isometry T_φ^m, $m \geq 1$. By (3.2) and (3.3) it follows that $E_m = T_{\varphi_1}^m T_{\varphi_1}^{*m}$ and $F_m = T_{\varphi_2}^m T_{\varphi_2}^{*m}$, and hence

$$\mathcal{R}(E_m) = \varphi_1^m H^2(\mathbb{D}^n) \text{ and } \mathcal{R}(F_m) = \varphi_2^m H^2(\mathbb{D}^n),$$

for all $m \geq 1$. Then, by Theorem 4.1, we have

$$\bigcap_{m\geq 0} \mathcal{R}(E_m) = \bigcap_{m \geq 0} \varphi_1^m H^2(\mathbb{D}^n) = \{0\},$$

and similarly $\bigcap_{m\geq 0} \mathcal{R}(F_m) = \{0\}$. Therefore, the unitary part, the shift part, and the co-shift part of the Halmos and Wallen model of T_φ are trivial (see [13, page 661] or [1]). Hence in this case, T_φ is a direct sum of truncated shifts. $\square$

Clearly, Corollary 5.1 immediately follows from the above result as well. Also, note that the Halmos and Wallen models of power partial isometries played an important role in the proof of the above theorem. We refer [1, 4, 10] for a more recent view point of power partial isometries.

Finally, summarizing our results from an operator theoretic point of view, we conclude the following: Let T_φ, $\varphi \in L^\infty(\mathbb{T}^n)$, be a partially isometric Toeplitz operator. Then the following hold:

(1) If $n = 1$, then T_φ is either an isometry, or a coisometry. This is due to Brown and Douglas. And, in view of Theorem 4.1, T_φ is either a shift, or a co-shift.
(2) If $n > 1$, then, up to unitary equivalence, T_φ is either a shift, or a co-shift, or a direct sum of truncated shifts.

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